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A Class of Fifth-Order Methods for Solving First Order Differential Equations

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Abstract

This paper proposes the derivation of continuous linear multistep methods for solving first-order initial value problems (IVPs) of ordinary differential equations (ODEs) with step number $k = 1$ with one-step with one off-step and $k = 2$ without off-step. The methods integrate some first-order initial value problems of ordinary differential equations. The discrete method developed from the continuous schemes using power series approximation as the basis function through interpolation and collocation approach adopts Taylor's series for the implementation. The methods are found to be consistent, zero stable, convergent and accurate. The experimental results showed to be convergent and efficient with when favourable performance when compared with some existing works through tested problems.

1. INTRODUCTION

We considered a First Order Ordinary Differential Equations with initial values as:

$$y' = f\left(x, y\right), y\left(x_0\right) = y_0 \tag{1}$$

f is a continuously differentiable function within an interval satisfying the existence and uniqueness of the solution (1). Ordinary Differential Equations are tools necessary in solving real-life problems. Various natural phenomena are modelled using ODEs which are applied to solve many problems. Thus, in recent years, ordinary differential equations have received a lot of attention. The collocation method is widely considered as a way of generating numerical solutions to the ordinary differential equation of the form (1). Collocation is a projection method for solving integral and differential equations, and the approximate solution is determined from the condition that the equation must be stratified at given points. It involves the determination of an approximate solution in a set of functions called the basis function. The usual way of solving (1) is to use a one-step explicit method such as Runge-Kutta of the same order of accuracy until enough values have been generated for a multistep method to take off. The problem with the linear multistep method is that they need help getting started which is encountered in single-step methods [1].

The help required in getting started is called a predictor. These starting values are called Predictors for (1) while the equation (1) is called corrector; hence the procedure is called the predictor-corrector method. The predictors are explicit, while the correctors are implicit methods. [4], proposed an order seven method implemented in predictor-corrector mode. The method used the corrector to give a continuous linear multistep method evaluated at some selected grid points to give a discrete linear multistep method of order seven for first ODEs. [5], developed a method using the block approach to handle the pronounced setback of the predictor-corrector method.

[3], proposed Taylor series approximation method to improve the setback usually faced with Predictor-Corrector and Block methods.
[6], developed a family of single-step continuous hybrid linear multistep methods (CHLMM) Taylor's series approximation as a simultaneous numerical integrator over non-overlapping intervals values and for the implementation of the methods for first-order ordinary differential equations. The method takes care of the tedious process of writing subroutine and developing corrector as noticed in the predictor-corrector. This approach provides the starting values, which makes it to be self-starting. [2], proposed hybrid block method of order six to yield five consistent finite difference schemes as simultaneous numerical integrators to form block method used to solve problems. This method provides a better global error estimate and a simpler form for further analytical work than the discrete ones.

Our interest in this paper is to develop some continuous multistep hybrid and non-hybrid methods with and collocated at all the grids and off grids points. This is generated from the continuous schemes to form the block methods for the implementation to give more accuracy in solving problems.

2. DERİVATİON OF METHODS

Consider the power series of the form:

$$y(x)=\sum_{j=0}^{k} a_j x^j \tag{2}$$

the first derivative of (2) gives:

$$y'(x)=\sum_{j=0}^{k} j a_j x^{j-1} \tag{3}$$

Putting (3) into (1) we obtain:

$$\sum_{j=0}^{k} j a_j x^{j-1} = f(x,y) \tag{4}$$

2.1 Derivation of Method 1

Interpolating (2) at $x=x_{n+k}, k=\frac{1}{3}$, and collocating (3) at $x=x_{n+\varphi}, \varphi=0,1,2$ gives a system of linear equation of the form:

$$AX=U \tag{5}$$

where,

$$A=\left[a_0,a_1,a_2,a_3\right]^T \; U=\left[f_n, f_{n+1}, f_{n+2}, y_n\right]^T$$

$$X=\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 \\ 0 & 1 & 2x_n & 3x_n^2 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 \end{bmatrix}$$

Solving (5) using Gaussian's elimination method, we have a continuous method of the form:

$$y(t)=\alpha_{\theta}(t)\,y_{n+\theta}+h\left[\sum_{\eta=0}^{k}\beta_{k}(t)\,f_{n+k}\right] \tag{6}$$

where $\theta=0, \theta = 0,\ \eta=0,1,2,\ \eta = 0,1,2,\ k=2, \kappa = 2,\ f_{n+k} = f\left(x_n+kh\right)$

Thus, using the transformation

$$t=\frac{x-x_{n+k}}{h} \tag{7}$$

we have a continuous method of the form:

$$\alpha_0(t)=1$$

$$\beta_0(t)=\frac{h}{36}\left[-76+2t+t^2+t^3\right]$$

$$\beta_1(t)=\frac{h}{12}\left[16+4t^2+3t^3-t^4\right] \tag{8}$$

$$\beta_2(t)=\frac{h}{3}\left[-87+2t+t^2+t^3\right]$$

Evaluating (8), at $t=2$ and $t=1$ which mean $x=x_{n+2}$ and $x=x_{n+1}$ which give a discrete method of the form:

$$y_{n+2}-y_n=\frac{h}{3}\left(f_{n+2}+4f_{n+1}+f_n\right) \tag{9}$$

$$y_{n+1}-y_n=\frac{h}{36}\left(-540f_{n+2}+156f_{n+1}-34f_n\right) \tag{10}$$

2.2 Development of Method 2

Interpolating (2) at $x=x_{n+u}, u=0,$ and collocating (3) at $x=x_{n+\varphi}, \varphi=0,\frac{1}{2},1$ gives a system of linear equation of the form:

$$AX=U \tag{11}$$

where,

$$A=\left[a_0,a_1,a_2,a_3\right]^T\ U=\left[f_n, f_{n+1}, f_{n+\frac{1}{2}}, y_n\right]^T$$

$$X=\begin{bmatrix}1 & x_n & x_n^2 & x_n^3\\ 0 & 1 & 2x_n & 3x_n^2\\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x_{n+\frac{1}{2}}^2\\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2\end{bmatrix}$$

Using Gaussian's elimination method for (11), we have a continuous method of the form:

$$y(t)=\alpha_{\theta}(t)\,y_{n+\theta}+h\left[\sum_{\eta=0}^{k}\beta_{k}(t)\,f_{n+k}\right] \tag{12}$$

Where

$\theta = 0,\ \eta = 0, \frac{1}{2}, 1,\ k = 1$

$f_{n+k} = f\left(x_n + kh\right)$ with the transformation:

$t = \frac{x - x_{n+k}}{h}$

we have a continuous method of the form:

$$\begin{aligned}
&\alpha_0(t) = 1 \\
&\beta_0(t) = \frac{h}{36}\left[-34 + 2t + t^3 + t^4\right] \\
&\beta_{1/2}(t) = \frac{h}{36}\left[-42 + t + 6t^2 + 5t^3\right] \\
&\beta_1(t) = \frac{h}{6}\left[1 + 4t^2 + 2t^3 + 2t^4\right]
\end{aligned} \tag{13}$$

at $t = 1$ and $t = \frac{1}{2}$ in (13), which mean $x = x_{n+1}$ and $x = x_{n+\frac{1}{2}}$ we have the block method to be the form:

$$y_{n+1} - y_n = \frac{h}{6}\left(f_{n+1} + 4f_{n+\frac{1}{2}} + f_n\right) \tag{14}$$

$$y_{n+\frac{1}{2}} - y_n = \frac{h}{48}\left(f_{n+1} - \frac{355}{6} f_{n+\frac{1}{2}} - \frac{171}{4} f_n\right) \tag{15}$$

3. ANALYSİS OF THE BASİC PROPERTİES OF THE BLOCK

The basic properties of the methods were examined. They are Zero Stability, Order of Accuracy and Error Constants and consistency.

3.1 Zero Stability

A Linear Multistep Method is said to be zero stable, if the roots of the first characteristic polynomial

$$\det\left[\lambda A^{(0)} - A^{(i)}\right] = 0$$

Satisfying $|\lambda| \le 1$ for the roots with $|\lambda| \le 1$ not exceed the order of the differential equations [8]. For our methods, we have

$$A = \left[z \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right] = 0$$

$$A = z^4\left(z - 1\right) = 0, z = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

Thus, we conclude that our methods are zero stable.

3.2 Order of Accuracy and Error Constant

A Block Linear Multistep Method is said to be of order p, if p is the largest positive integer for which

$C_0 = C_1 = C_2 = \cdots, C_{p+1} = 0$ and $C_{P+1} \neq 0$ is called the error constant and $C_{p+1}h^{p+1}y^{p+1}(x_n)$ is the principal Local Truncation Error (LTE) at the point x_n. Expanding our methods by Taylor series expansion, comparing coefficients equating to zero using the method adopted by [6], we have the two block methods of be of uniform order $p = 5$ with error constants of

$$C_{p+1} = \left[-\frac{1}{90}, -\frac{1}{126}\right]^T \text{ and } C_{p+1} = \left[-\frac{1}{2880}, -\frac{1}{3365}\right]^T$$

3.3 Consistency

According to [6], the methods are consistent if the order of the method is $p \geq 1$. Since, the Methods are $p = 4 \geq 1$ satisfying the condition. We conclude that the methods are consistent.

3.4 Convergence

According to [7], for a method to be convergence, it must be consistent and zero stable. Since these conditions are satisfied, then the methods are said to be convergent.

4. NUMERİCAL EXAMPLES

We verify the validity and performance of the methods with some test problems.

Problem 1

$$y' + y = 0, \qquad y(0) = 1, \ h = 0.1$$

Exact Solution: $y(x) = e^{-x}$

Problem 2

$$y' - 2xy = 0, \quad y(0) = 1, \ h = 0.1$$

Exact Solution: $y(x) = e^{x^2}$

5. NUMERİCAL RESULTS

Table 1: Numerical Result for Method 1 to solve Problem 1

x	Exact Solution	Numerical Solution	Error
0.1	0.90483741803596	0.90483800989126	5.9185E − 09
0.2	0.81873075307798	0.81873182414398	1.0710E − 08
0.3	0.74081822068172	0.74081967439308	1.4537E − 08
0.4	0.67032004603564	0.67032179986613	1.7538E − 08
0.5	0.60653065971263	0.60653264337760	1.9836E − 08
0.6	0.54881163609403	0.54881378996787	2.1538E − 08
0.7	0.49658530379141	0.49658757751541	2.2737E − 08
0.8	0.44932896411722	0.44933131537576	2.3512E − 08
0.9	0.40656965974060	0.40657205318643	2.3934E − 08
1.0	0.36787944117144	0.36788184748261	2.4063E − 08

Table 2: Numerical Result for Method 2 to solve Problem 1

x	Exact Solution	Numerical Solution	Error
0.1	0.90483741803596	0.90483756565329	1.4761E − 10
0.2	0.81873075307798	0.81873102021737	2.6713E − 10
0.3	0.74081822068172	0.74081858325831	3.6257E − 09
0.4	0.67032004603564	0.67032048346617	3.6257E − 09
0.5	0.60653065971263	0.60653115446706	3.6257E − 09
0.6	0.54881163609403	0.54881217330085	3.6257E − 09

0.7	0.49658530379141	0.49658587089043	3.6257E − 09
0.8	0.44932896411722	0.44932955055432	3.6257E − 09
0.9	0.40656965974060	0.40657025669965	3.6257E − 09
1.0	0.36787944117144	0.36788004133915	5.9695E − 08

Table 3: Numerical Result for Method 1 to solve Problem 2

x	Exact Solution	Numerical Solution	Error
0.1	1.01005016708417	1.01005003025771	1.368E − 09
0.2	1.04081077419239	1.04081020290595	5.712E − 09
0.3	1.09417428370521	1.09417287622282	1.407E − 09
0.4	1.17351087099181	1.17350802531083	2.845E − 08
0.5	1.28402541668774	1.28402018730910	5.229E − 08
0.6	1.43332941456034	1.43332028673732	9.127E − 08
0.7	1.63231621995538	1.63230074357230	1.547E − 07
0.8	1.89648087930495	1.89645506523525	2.581E − 07
0.9	2.24790798667646	2.24786529966716	4.268E − 07
1.0	2.71828182845903	2.71821148946015	7.033E − 07

Table 4: Numerical Result for Method 2 to solve Problem 2

x	Exact Solution	Numerical Solution	Error
0.1	1.01005016708417	1.01005013337654	3.370E − 10
0.2	1.04081077419239	1.04081063230877	1.418E − 09
0.3	1.09417428370521	1.09417393306249	3.506E − 09
0.4	1.17351087099181	1.17351016077859	7.102E − 09
0.5	1.28402541668774	1.28402410991864	1.306E − 08
0.6	1.43332941456034	1.43332713138770	2.283E − 08
0.7	1.63231621995538	1.63231234569132	3.874E − 08
0.8	1.89648087930496	1.89647441271315	6.466E − 08
0.9	2.24790798667649	2.24789728676892	1.069E − 07
1.0	2.71828182845908	2.71826418752465	1.764E − 07

Table 5: Comparison of Results for Problem 1

x	ERMD 1	ERMD 2	Error in [1]
0.1	1.368E − 09	3.370E − 10	8.196404E − 008
0.2	5.712E − 09	1.418E − 09	1.483283E − 007
0.3	1.407E − 09	3.506E − 09	3.826926E − 008
0.4	2.845E − 08	7.102E − 09	3.563031E − 008
0.5	5.229E − 08	1.306E − 08	2.728729E − 007
0.6	9.127E − 08	2.283E − 08	2.961309E − 007
0.7	1.547E − 07	3.874E − 08	5.594595E − 007
0.8	2.581E − 07	6.466E − 08	5.773206E − 007
0.9	4.268E − 07	1.069E − 07	8.492640E − 007
1.0	7.033E − 07	1.764E − 07	8.439141E − 007

6. DİSCUSSİON OF RESULTS

Table 1 - Table 4 shows the results generated from methods 1 and 2 using it to solve problems 1 and 2. The results in Table 1 - Table 4 shows that method 2 with a single step and one off-step point perform better than the two-step method in term of accuracy and computational time. Our methods produced better performance when compared with [1], using it to solve the same

problem as presented in table 5. Likewise, Table 6 shows the performance of the two methods in terms of accuracy comparing with [1] for solving problem 2. Thus, the proposed methods have demonstrated better performance and efficiency in terms of accuracy [1].

7. CONCLUSİON

This research has presented a class of continuous methods for the numerical integration of first-order initial value problems. It is noteworthy that the results generated from the methods are significantly accurate and efficient having compared to other existing works. The proposed methods will be effective in solving first-order differential equations.

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CONFLICTS OF INTEREST
No conflict of interest was declared by the authors.

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